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ON SAINT-VENANT TYPE CONDITIONS IN THE THEORY OF PIEZOELASTIC SHELLS*

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Saint-Venant type conditions extended to piezoelasticity are formulated. It is shown that one electrical condition is added to the well-known Saint-Venant mechanical conditions for piezoceramic shells with non-electrodized face surfaces. The Saint-Venant conditions accepted in elasticity theory remain true for shells with electrodized face surfaces.

The complete state of stress and strain of a non-electric elastic shell is comprised of a deeply penetrating internal state of stress and strain described by the equations of shell theory, and of boundary layers localized near the edges. In formulating the boundary conditions for the internal state of stress and strain and the boundary layers, an important part is played by the Saint-Venant principle /1/, which is as follows as applied to elastic shells: if stresses are given arbitrarily on the edge of a shell, then non-selfequilibrated edge effects will generate a deeply penetrating solution and should be taken into account when analysing the state of stress and strain, while the part of the edge load not selfequilibrated over the thickness will cause a stress and strain state that will damp rapidly at the edge and is taken into account in analysing the boundary layer.

In the case of piezoelastic shells, both electrical and mechanical quantities occur in the complete system of equations. Consequently, the question arises of what conditions of Saint-Venant type should the mechanical and electrical edge load satisfy. To answer this question, following /2/, we find a solution of the boundary layer problems and we clarify, in passing, what requirements the edge load should be subjected to in order for the boundary layer solution to have the necessary damping. That part of the load which does not satisfy these conditions should be taken into account in analysing the internal electroelastic state of the shell.

1. We select a system of tri-orthogonal coordinates as follows: curvilinear coordinates α_1 and α_2-lines of curvature of the middle surface, and γ-lines orthogonal to them.

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The piezoceramic shell equations and their corresponding boundary-layer equations depend substantially on the directions of preliminary piezoceramic polarization. To be specific, we consider the boundary-layer at the edge $\alpha_1 = \alpha_{10}$ in a piezoceramic shell with preliminary polarization along the α_2-lines. As is customary in the asymptotic method, we stretch the coordinates in the direction of greatest variability of the quantities (in directions orthogonal to the edge and in thickness)

$$A_1(\alpha_1 - \alpha_{10}) = h\xi, \quad \gamma = h\zeta \tag{1}$$

The notation used here is identical with that used in /3/.

As is shown in /3/, the boundary-layer analysis in the initial approximation reduces to solving plane and antiplane problems for a piezoceramic half-strip with homogeneous conditions on the face surfaces.

We assume that an arbitrary mechanical and electrical load

$$S_{11} = f_1(\zeta), \quad S_{13} = f_2(\zeta) \tag{2}$$

$$S_{12} = f_3(\zeta), \quad \varphi = \psi_1(\zeta), \quad D_1 = \psi_2(\zeta) \tag{3}$$

is given on the edge $\xi = 0$ of the half-strip.

Homogeneous conditions

$$S_{23} = 0, \quad D_3 = 0, \quad S_{13} = 0, \quad S_{33} = 0 \tag{4}$$

are given on the non-electrodized face surfaces of the half-strip $\zeta = 1$ and $\zeta = -1$.

Here S_{ij} are stresses, φ is the electrical potential, D_1 is the electric induction vector component normal to the edge surface, and D_3 is the electric induction vector component normal to the face surface.

Apart from physical constants, the equations of a plane piezoelastic boundary-layer at the edge $\alpha_1 = \alpha_{10}$ agree with the equations of a plane boundary-layer in elasticity theory; consequently, the usual Saint-Venant damping conditions /2/ hold for a plane piezoelastic boundary-layer

$$\int_{-h}^{+h} S_{11}(\alpha_{10})\, d\gamma = 0, \quad \int_{-h}^{+h} S_{13}(\alpha_{10})\, d\gamma = 0, \quad \int_{-h}^{+h} \gamma S_{11}(\alpha_{10})\, d\gamma = 0 \tag{5}$$

Let us obtain the damping conditions for an antiplane boundary-layer. The equations of the antiplane piezoelastic problem have the following form in the dimensionless coordinates (1):

$$\frac{\partial S_{21}}{\partial \xi} + \frac{\partial S_{23}}{\partial \zeta} = 0, \quad \frac{\partial D_1}{\partial \xi} + \frac{\partial D_3}{\partial \zeta} = 0 \tag{6}$$

$$\frac{\partial v}{\partial \xi} = s_{44}{}^E S_{21} - d_{15}\frac{\partial \varphi}{\partial \xi}, \quad \frac{\partial v}{\partial \zeta} = s_{66}{}^E S_{23} - d_{15}\frac{\partial \varphi}{\partial \zeta}$$

$$D_1 = d_{15}S_{21} - \varepsilon_{11}{}^T\frac{\partial \varphi}{\partial \xi}, \quad D_3 = d_{15}S_{23} - \varepsilon_{11}{}^T\frac{\partial \varphi}{\partial \zeta}$$

This system can be reduced to two equations in the unknowns v and φ

$$k^2\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \zeta^2} = (1 - k^2)\, d_{15}\frac{\partial^2 \varphi}{\partial \xi^2}, \quad \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \zeta^2} = 0, \tag{7}$$

$$k^2 = s_{66}{}^4/s_{44}{}^E$$

When solving system (7) we satisfy the first two conditions of (4) on the face surfaces of the half-strip and conditions (3) on the edge, where the second condition of (3) must be satisfied on the edge by electrodes, and the third condition of (3) on the non-electrodized edge.

We perform a Laplace integral transform in the variable ξ in system (7), whereupon we obtain

$$U = \int_0^\infty v(\xi, \zeta)\, e^{-p\xi}\, d\xi, \quad \Phi = \int_0^\infty \varphi(\xi, \zeta)\, e^{-p\xi}\, d\xi \tag{8}$$

$$d^2U/d\zeta^2 + k^2p^2U = k\Omega_2 - \Omega_1 + (1 - k^2)\, d_{15}p\Phi$$

$$d^2\Phi/d\zeta^2 + p^2\Phi = \Omega_1$$

$$\Omega_1 = \left(\frac{\partial \varphi}{\partial \xi} + p\varphi\right)_{\xi=0}, \quad \Omega_2 = k\left(\frac{\partial v}{\partial \xi} + pv\right)_{\xi=0} + kd_{15}\Omega_1$$

In the new notation, the conditions on the non-electrodized face surfaces $\zeta = 1$, $\zeta = -1$ are written thus:

$$dU/d\zeta = 0, \quad d\Phi/d\zeta = 0 \tag{9}$$

We integrate (8). Their solutions have the following form:

$$\Phi = C_1 \cos p\zeta + C_2 \sin p\zeta + a_1(p, \zeta) \cos p\zeta + b_1(p, \zeta) \sin p\zeta \tag{10}$$

$$U = C_3 \cos kp\zeta + C_4 \sin kp\zeta - d_{15} \{[C_1 + a_1(p, \zeta)] \cos p\zeta + [C_2 + b_1(p, \zeta)] \sin p\zeta\} + a_2(p, \zeta) \cos kp\zeta + b_2(p, \zeta) \sin kp\zeta$$

$$a_1(p,\zeta) = -\frac{1}{p}\int_{\zeta_0}^{\zeta} \Omega_1 \sin p\zeta\, d\zeta, \quad b_1(p,\zeta) = \frac{1}{p}\int_{\zeta_1}^{\zeta} \Omega_1 \cos p\zeta\, d\zeta$$

$$a_2(p,\zeta) = -\frac{1}{p}\int_{\zeta_0}^{\zeta} \Omega_2 \sin kp\zeta\, d\zeta, \quad b_2(p,\zeta) = \frac{1}{p}\int_{\zeta_1}^{\zeta} \Omega_2 \cos kp\zeta\, d\zeta$$

We determine the constants $C_1, \ldots, C_4$ from conditions (9).

We assume that the functions (3) given on the edge are even functions of ζ. Then if we set $\zeta_0 = -1,\ \zeta_1 = 0,$ we obtain

$$C_2 = C_4 = 0, \quad a_1 = a_2 = 0, \quad C_1 = b_1(p, 1)\frac{\cos p}{\sin p} \tag{11}$$
$$C_3 = \frac{1}{k \sin kp}[-d_{15} b_1(p, 1) \cos p + d_{15} C_1 \sin p + k b_2(p, 1) \cos kp]$$

We substitute (11) into (10), then we find the residues of the functions U and Φ and applying the inversion theorem obtain the final formulas for v and φ. For the formulas for v and φ not to contain terms that grow according to a power law, we equate the residues to zero at the points $p = 0$, whereupon we obtain Stain-Venant type damping conditions

$$\int_{-h}^{h} S_{12}(\alpha_{10})\, d\gamma = 0, \quad \int_{-h}^{+h} \varphi(\alpha_{10})\, d\gamma = 0, \quad \int_{-h}^{+h} D_1(\alpha_{10})\, d\gamma = 0 \tag{12}$$

The second of conditions (12) should be satisfied on the electrodized edge, and the third on the non-electrodized edge.

Let the functions (3) given on the edge be odd functions of ζ. Then by setting $\zeta_0 = 0,$ $\zeta_1 = -1,$ we obtain

$$C_1 = C_3 = 0, \quad b_1(p, 1) = b_2(p, 1) = 0$$
$$C_2 = \frac{a_1(p, 1) \sin p}{\cos p}, \quad C_4 = \frac{a_2(p, 1) \sin kp}{\cos kp}$$

It is seen from these last formulas that the functions U and Φ have residues only at points where the equalities $\cos p_n = 0,\ \cos kp_m = 0$ are satisfied. The roots of these equations are the numbers $p_n, -p_n, p_m, -p_m,$ where

$$p_n = \frac{\pi}{2}(2n - 1), \quad p_m = \frac{\pi}{2k}(2m - 1) \quad (n, m = 1, 2, 3 \ldots)$$

The desired functions have no residues at the points $p = 0$, consequently, the solution of the antiplane boundary-layer will be damped for any edge load (3) odd in ζ.

Using the inversion theorem, we find

$$\varphi = \sum_{n=1}^{\infty} (\mathrm{res}_{p_n} \Phi e^{p_n \xi} + \mathrm{res}_{-p_n} \Phi e^{-p_n \xi})$$
$$v = \sum_{n=1}^{\infty} (\mathrm{res}_{p_n} U e^{p_n \xi} + \mathrm{res}_{-p_n} U e^{-p_n \xi}) + \sum_{m=1}^{\infty} (\mathrm{res}_{p_m} U e^{p_m \xi} + \mathrm{res}_{-p_m} U e^{-p_m \xi})$$

The residues at the points p_n and p_m yeild solutions that increase with distance from the edge, We hence equate them to zero (the solution obtained is sufficiently arbitrary to satisfy these conditions), and we consequently obtain

$$a_1(p_n, 1) = -\frac{1}{p_n}\int_0^1 \left(\frac{\partial \varphi}{\partial \xi} + p_n \varphi\right)_{\xi=0} \sin p_n\zeta\, d\zeta = 0 \tag{13}$$
$$a_2(p_m, 1) = -\frac{k}{p_m}\int_0^1 \left[\frac{\partial v}{\partial \xi} + d_{15}\frac{\partial \varphi}{\partial \xi} + p_m(v + d_{15}\varphi)\right]_{\xi=0} \sin kp_m\zeta\, d\zeta = 0$$

Using (13) for φ and v we can write the exponentially damped solution

$$\varphi = \sum_{n=1}^{\infty} \mathrm{res}_{-p_n} \Phi e^{-p_n \xi}, \quad v = \sum_{n=1}^{\infty} \mathrm{res}_{-p_n} U e^{-p_n \xi} + \sum_{m=1}^{\infty} \mathrm{res}_{-p_m} U e^{-p_m \xi}$$

Here

$$\mathrm{res}_{-p_n} \Phi = a_1(-p_n, 1) \sin p_n\zeta \tag{14}$$
$$\mathrm{res}_{-p_n} U = k^2 a_2(-p_m, 1) \sin kp_m\zeta$$

$$\mathrm{res}_{-p_n} U = -d_{15} a_1 (-p_n, 1) \sin p_n \zeta$$

If the value of the electric potential φ is given on the edge, then by eliminating the unknown functions $(\partial\varphi/\partial\xi)_{\xi=0}$ and $(v + d_{15}\varphi)_{\xi=0}$ in (14), taking (13) into account, we obtain the final formulas

$$\varphi = \sum_{n=1}^{\infty} c^{(n)} \sin p_n \zeta e^{-p_n \xi}, \quad v = -d_{15}\varphi + \sum_{m=1}^{\infty} b^{(m)} \sin p_m \zeta e^{-p_m \xi}$$

$$c^{(n)} = 2 \int_0^1 \psi_1(\zeta) \sin p_n \zeta \, d\zeta, \quad b^{(m)} = \frac{2s_{44}{}^E}{p_n} \int_0^{\infty} f_3(\zeta) \sin k p_m \zeta \, d\zeta$$

If the electric induction vector component normal to the edge surface D_1 is given on the edge, then the formula for $c^{(n)}$ should be replaced by

$$c^{(n)} = -\frac{2}{p_n \varepsilon_{11}{}^T} \int_0^1 [d_{15} f_3(\zeta) - \psi_2(\zeta)] \sin p_n \zeta \, d\zeta$$

Thus, for piezoceramic shells with non-electrodized face surfaces the Saint-Venant generalized conditions for stresses given at the edge are conserved in the same form as in elasticity theory (non-selfequilibrated edge effects generate a solution that damps exponentially at the edge), while for electrical quantities given at the edge the following damping conditions hold:

$$\int_{-h}^{+h} \varphi(\alpha_{10}) \, d\gamma = 0$$

on the electrodized edge and

$$\int_{-h}^{+h} D_1(\alpha_{10}) \, d\gamma = 0$$

on the non-electrodized edge.

These conditions were obtained on the edge $\alpha_1 = \alpha_{10}$ for a shell pre-polarized along the α_2-line. It can be shown in an analogous manner that the damping conditions obtained hold even on the edge $\alpha_2 = \alpha_{20}$ and on the edge of a shell with thickness polarization. Solutions of the corresponding boundary layer problems are obtained but are not presented here because of the awkwardness of the computations since the governing equation of the plane electroelastic problem is of sixth order in these cases (fourth order in elasticity) and the algebraic aspect of the problem becomes much more complicated.

If the shell face surfaces are electrodized, then the second condition in (4) should be replaced by the condition $\varphi = 0$ for $\gamma = h$, $\gamma = -h$ and by using solution (10) it can be shown that the damping condtions which should be imposed on the edge load on this case will agree completely with the Saint-Venant conditions in elasticity theory. Any edge electrical load, either selfequilibrated or not selfequilibrated will cause an exponentially damped electroelastic state with distance from the edge. This result agrees completely with the fact that the equations of the theory of piezoceramic shells with electrodized face surfaces agree with the equations for non-electric shells apart from constant coefficients and have no arbitrariness to satisfy the electrical conditions on the shell edges /3, 4/.

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